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Göteborg ITP 99-06

Open group transformations

Igor Batalin[1] and Robert Marnelius[2]

Institute of Theoretical Physics
Chalmers University of Technology
Göteborg University
S-412 96 Göteborg, Sweden

Abstract

Open groups whose generators are in arbitrary involutions may be quantized within a ghost extended framework in terms of a nilpotent BFV-BRST charge operator. Previously we have shown that generalized quantum Maurer-Cartan equations for arbitrary open groups may be extracted from the quantum connection operators and that they also follow from a simple quantum master equation involving an extended nilpotent BFV-BRST charge and a master charge. Here we give further details of these results. In addition we establish the general structure of the solutions of the quantum master equation. We also construct an extended formulation whose properties are determined by the extended BRST charge in the master equation.

[1]On leave of absence from P.N.Lebedev Physical Institute, 117924 Moscow, Russia
E-mail: batalin@td.lpi.ac.ru.

[2]E-mail: tferm@fy.chalmers.se.

1 Introduction

Lie groups are dominant continuous transformations in theoretical physics. However, there are important models in which more general kinds of continuous transformations enter. For instance, the gauge transformations in field theories like gravity, supergravity and p-branes are not Lie groups. Open groups whose generators are in arbitrary involutions are very difficult to handle. The only framework in which they can be treated systematically in quantum theory is within a ghost extended BRST frame. This is consistent when the group algebra may be imbedded in a nilpotent BRST charge following the BFV-prescription [1]. Representations of open groups are then naturally classified in terms of ranks [1]. Ordinary nonabelian Lie groups are of rank 1. The gauge group of supergravity has rank 2 [2], and that of p-branes rank p [3].

In [4] we investigated the properties of group transformed operators for arbitrary open groups within the BFV-BRST-scheme [1]. By means of a particular representation of the operator connections in the Lie equations for the transformed operators, we arrived at generalized Maurer-Cartan equations expressed in terms of the quantum antibracket introduced in [5]. Furthermore, we presented a new type of quantum master equation, also expressed in terms of the quantum antibracket, which we proposed to encode these generalized Maurer-Cartan equations. Here we give further details of these results. In addition we establish the general structure of the solutions of the quantum master equation. We also construct an extended formulation whose properties are determined by the extended BRST charge in the master equation.

2 Quantizing arbitrary involutions

Consider a dynamical system with a finite number of degrees of freedom. Let the phase space (or the symplectic manifold), Γ, for this system be spanned by the coordinates z^A. Thus, $\{z^A, z^B\}$ is an invertable matrix in terms of the Poisson bracket. (Locally z^A consists of canonical conjugate pairs: q^i, p_i.) The generators of the open group under consideration are functions on Γ denoted by $\theta_a(z)$ with Grassmann parities $\varepsilon(\theta_a) \equiv \varepsilon_a (= 0, 1)$ satisfying the Lie algebra

$$\{\theta_a(z), \theta_b(z)\} = U_{ab}{}^c(z)\theta_c(z). \tag{2.1}$$

As is well-known it is extremely difficult to construct a quantum theory in which the corresponding commutator algebra to (2.1) is integrable when the structure coefficients $U_{ab}{}^c$ are functions on Γ. The canonical procedure which makes this possible is to first extend the phase space Γ by ghost variables and then imbed the algebra (2.1) in one odd real function Ω satisfying $\{\Omega, \Omega\} = 0$ in terms of the extended Poisson bracket. Ω is the BFV-BRST charge [1]. The corresponding quantum theory is now consistent if the corresponding odd, hermitian operator Ω is nilpotent, *i.e.* $\Omega^2 = 0$. For a finite number of degrees of freedom such a solution always exists and is of the form [6] (N is the rank of the theory. θ_a are assumed to be hermitian operators.)

$$\Omega = \sum_{i=0}^{N} \Omega_i, \tag{2.2}$$

$$\Omega_0 \equiv \mathcal{C}^a\theta_a(z), \quad \Omega_i \equiv \Omega^{b_i\cdots b_1}_{a_1\cdots a_{i+1}}(z)(\mathcal{P}_{b_1}\cdots\mathcal{P}_{b_i}\mathcal{C}^{a_{i+1}}\cdots\mathcal{C}^{a_1})_{Weyl}, \quad i=1,\ldots,N, \tag{2.3}$$

where $\mathcal{C}^a$, $\mathcal{P}_a$ are the ghost operators with Grassmann parities $\varepsilon(\mathcal{C}^a)=\varepsilon(\mathcal{P}_a)=\varepsilon_a+1$, satisfying the properties

$$[\mathcal{C}^a,\mathcal{P}_b]=i\hbar\delta^a_b, \quad (\mathcal{C}^a)^\dagger=\mathcal{C}^a, \quad \mathcal{P}^\dagger_a=-(-1)^{\varepsilon_a}\mathcal{P}_a. \tag{2.4}$$

All commutators are from now on graded commutators defined by

$$[A,B]\equiv AB-BA(-1)^{\varepsilon_A\varepsilon_B}, \tag{2.5}$$

where ε_A and ε_B are the Grassmann parities of the operators A and B respectively. In (2.3) the ghost operators are Weyl ordered which means that Ω_i are all hermitian. Ω determines the precise form of the quantum counterpart of the algebra (2.1). A convenient form of this algebra is obtained if we rewrite Ω in the following $\mathcal{CP}$-ordered form [6]

$$\Omega=\sum_{i=0}^{N}\Omega'_i, \quad \Omega'_0\equiv\mathcal{C}^a\theta'_a(z),$$
$$\Omega'_i\equiv\mathcal{C}^{a_{i+1}}\cdots\mathcal{C}^{a_1}\Omega'^{b_i\cdots b_1}_{a_1\cdots a_{i+1}}(z)\mathcal{P}_{b_1}\cdots\mathcal{P}_{b_i}, \quad i=1,\ldots,N. \tag{2.6}$$

The nilpotency of Ω requires then the commutator algebra

$$[\theta'_a(z),\theta'_b(z)]=i\hbar {U'}_{ab}^{\ \ c}(z)\theta'_c(z), \tag{2.7}$$

where the structure operators ${U'}_{ab}^{\ \ c}(z)$ are given by

$${U'}_{ab}^{\ \ c}(z)=2(-1)^{\varepsilon_b+\varepsilon_c}{\Omega'}_{ab}^{\ \ c}(z). \tag{2.8}$$

In terms of the coefficient operators in (2.3), ${\Omega'}_{ab}^{\ \ c}(z)$ and $\theta'_a(z)$ are given by

$${\Omega'}^c_{ab}(z)=\Omega^c_{ab}(z)+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^n(n+1)(n+2)!\,\Omega^{a_n\cdots a_1 c}_{aba_1\cdots a_n}(z)(-1)^{\sum_{k=1}^n\varepsilon_{a_k}},$$
$$\theta'_a(z)=\theta_a(z)+\sum_{n=1}^{\infty}\left(\frac{i\hbar}{2}\right)^n(n+1)!\,\Omega^{a_n\cdots a_1}_{aa_1\cdots a_n}(z)(-1)^{\sum_{k=1}^n\varepsilon_{a_k}}, \tag{2.9}$$

which shows that $\theta'_a(z)$ in general are different from $\theta_a(z)$ and in general not even hermitian. However, the main point here is that Ω through *e.g.* (2.7) represents the quantum counterpart of (2.1).

All operators and states may be decomposed into operators and states with definite ghost numbers. The ghost number g is defined by

$$G|A\rangle_g=i\hbar g|A\rangle_g, \quad [G,A_g]=i\hbar gA_g, \tag{2.10}$$

where G is the hermitian ghost charge operator defined by

$$G\equiv-\frac{1}{2}\left(\mathcal{P}_a\mathcal{C}^a-\mathcal{C}^a\mathcal{P}_a(-1)^{\varepsilon_a}\right). \tag{2.11}$$

One may notice that Ω in (2.2) has ghost number one, *i.e.* $[G,\Omega]=i\hbar\Omega$.

In a BRST-quantization which requires us to solve the BRST cohomology resulting from the BRST condition

$$\Omega|phys\rangle = 0, \tag{2.12}$$

the original generators θ_a in (2.1) are constraint variables which generate gauge transformations. Within the BRST quantization the gauge generators have a BRST exact form, *i.e.* they are of the form $[\Omega, \rho]$. The natural gauge generators are $[\Omega, \mathcal{P}_a]$. However, $[\Omega, \mathcal{P}_a]$ do not satisfy a closed algebra for higher rank theories ($N = 2$ and higher). On the other hand, $[\Omega, \mathcal{P}_a]$ and $\mathcal{P}_a$ which constitute BRST-doublets do always satisfy a closed algebra.

3 Finite group transformations for open groups

We want now to integrate the quantum involution (2.1) encoded in Ω as represented by *e.g.* (2.7). We consider therefore the Lie equations for the group transformed states and operators given by (cf [4])

$$\langle A(\phi)| \overleftarrow{D}_a \equiv \langle A(\phi)| \left(\overleftarrow{\partial}_a - (i\hbar)^{-1} Y_a(\phi) \right) = 0, \tag{3.1}$$

$$A(\phi) \overleftarrow{\nabla}_a \equiv A(\phi) \overleftarrow{\partial}_a - (i\hbar)^{-1}[A(\phi), Y_a(\phi)] = 0, \tag{3.2}$$

where ∂_a is a derivative with respect to the group parameter ϕ^a, $\varepsilon(\phi^a) = \varepsilon_a$. The connection operator Y_a, which depends on ϕ^a, must satisfy integrability conditions

$$Y_a \overleftarrow{\partial}_b - Y_b \overleftarrow{\partial}_a (-1)^{\varepsilon_a \varepsilon_b} = (i\hbar)^{-1}[Y_a, Y_b]. \tag{3.3}$$

A formal solution is

$$Y_a(\phi) \equiv i\hbar U(\phi) \left(U^{-1}(\phi) \overleftarrow{\partial}_a \right), \tag{3.4}$$

which means that the group transformed states and operators are of the form

$$\langle A(\phi)| = \langle A|U^{-1}(\phi), \quad A(\phi) = U(\phi) A U^{-1}(\phi), \tag{3.5}$$

where $U(\phi)$ is a finite group element. $U(\phi)$ must be an even operator and it is natural to require it to satisfy the conditions

$$U(0) = \mathbf{1}, \quad [\Omega, U(\phi)] = 0, \quad [G, U(\phi)] = 0. \tag{3.6}$$

For an exponential representation this implies

$$U(\phi) = \exp\{\frac{i}{\hbar} F(\phi)\}, \quad F(0) = 0, \quad [\Omega, F(\phi)] = 0, \quad [G, F(\phi)] = 0. \tag{3.7}$$

The first condition is just a choice of parametrization. The second condition is necessary in order to be consistent with a BRST quantization defined by (2.12). The last condition makes the group transformed state to have the same ghost number as the original one. Since in a BRST quantization the gauge generators are represented by BRST exact operators it is natural to expect $U(\phi)$ to have the form

$$U(\phi) = \exp\{-(i\hbar)^{-2}[\Omega, \rho(\phi)]\}, \tag{3.8}$$

where $\rho(\phi)$ has ghost number minus one. It could *e.g.* be $\rho(\phi) \propto \mathcal{P}_a\phi^a$. ((3.7) requires $[\Omega, \rho(0)] = 0$.) It is of course natural for $U(\phi)$ to be a unitary operator.

Let us now go back to the Lie equations (3.1) and (3.2). The conditions (3.6) applied to the formal expression (3.4) for the connection operator Y_a implies that

$$[\Omega, Y_a] = 0, \quad [G, Y_a] = 0. \tag{3.9}$$

In terms of the exponential representation (3.7) we have also

$$Y_a(\phi) = \int_0^1 d\alpha \exp\{\frac{i}{\hbar}\alpha F(\phi)\}\left(F(\phi)\overleftarrow{\partial}_a\right)\exp\{-\frac{i}{\hbar}\alpha F(\phi)\}. \tag{3.10}$$

In particular we have $Y_a(0) = F(\phi)\overleftarrow{\partial}_a\Big|_{\phi=0}$. From the natural representation (3.8) with $\rho(\phi) = \mathcal{P}_a\phi^a$ we have then $Y_a(0) = [\Omega, \mathcal{P}_a]$ which are the natural gauge generators.

The representation (3.8) yields the general property

$$Y_a(\phi) = (i\hbar)^{-1}[\Omega, \Omega_a(\phi)], \quad \varepsilon(\Omega_a) = \varepsilon_a + 1, \tag{3.11}$$

where Ω_a has ghost number minus one. In fact, (3.8) suggests that

$$\Omega_a(\phi) = \int_0^1 d\alpha \exp\{-(i\hbar)^{-2}\alpha[\Omega, \rho(\phi)]\}\left(\rho(\phi)\overleftarrow{\partial}_a\right)\exp\{(i\hbar)^{-2}\alpha[\Omega, \rho(\phi)]\}. \tag{3.12}$$

The simplest ansatz $\rho(\phi) = \mathcal{P}_a\phi^a$ implies that $\Omega_a(\phi)$ has the form

$$\Omega_a(\phi) = \lambda_a^b(\phi)\mathcal{P}_b + \{\text{possible ghost dependent terms}\}, \quad \lambda_a^b(0) = \delta_a^b, \tag{3.13}$$

where $\lambda_a^b(\phi)$ are operators in general. Notice that $\Omega_a(\phi)$ is only defined up to BRST invariant operators from a given $Y_a(\phi)$ exactly like $\rho(\phi)$ from a given $F(\phi)$. However, since the only BRST invariant operators with negative ghost numbers are BRST exact ones the arbitrariness is

$$\Omega_a(\phi) \to \Omega_a(\phi) + (i\hbar)^{-1}[\Omega, K_a(\phi)], \quad \rho(\phi) \to \rho(\phi) + (i\hbar)^{-1}[\Omega, \kappa(\phi)], \tag{3.14}$$

where $K_a(\phi)$ and $\kappa(\phi)$ have ghost number minus two. If the relation (3.11) is applied to the exponential representation (3.7) then we get from (3.10)

$$F(\phi)\overleftarrow{\partial}_a = (i\hbar)^{-1}[\Omega, G_a(\phi)] \;\Rightarrow\; F(\phi) = F(0) + (i\hbar)^{-1}[\Omega, \rho(\phi)], \tag{3.15}$$

where

$$\rho(\phi) = \int_0^1 d\alpha G_a(\alpha\phi)\phi^a + (i\hbar)^{-1}[\Omega, \kappa(\phi)]. \tag{3.16}$$

(Contract the first relation in (3.15) with ϕ^a and use the argument in appendix B.) The operators $G_a(\phi)$ must satisfy the integrability conditions

$$G_a(\phi)\overleftarrow{\partial}_b - G_b(\phi)\overleftarrow{\partial}_a(-1)^{\varepsilon_a\varepsilon_b} = (i\hbar)^{-1}[\Omega, G_{ab}(\phi)], \tag{3.17}$$

where $G_{ab}(\phi)$ in turn must satisfy the integrability conditions following from (3.17) and so on. Thus, if $F(0) = 0$ as required in (3.7) then the representation (3.8) follows from (3.11). Since the original algebra (2.1) is the algebra of the gauge group in a BRST quantization and this is the algebra we want to integrate the representation (3.11) of the connection operator $Y_a(\phi)$ is the natural one. The relation (3.11) was also the starting point in [4]. There we noticed that the integrability conditions (3.3) lead to a whole set of integrability conditions for $\Omega_a(\phi)$. The representation (3.11) inserted into (3.3) implies

$$[\Omega, \Omega_a\overleftarrow{\partial}_b - \Omega_b\overleftarrow{\partial}_a(-1)^{\varepsilon_a\varepsilon_b} - (i\hbar)^{-2}(\Omega_a, \Omega_b)_\Omega] = 0, \tag{3.18}$$

where we have introduced the quantum antibracket defined in (A.1) in appendix A. (They were introduced in [5, 7] and the relevant formulas are given in appendix A.) Since the right entry has ghost number minus one, it is zero up to a BRST exact operator. We have therefore [4]

$$\Omega_a\overleftarrow{\partial}_b - \Omega_b\overleftarrow{\partial}_a(-1)^{\varepsilon_a\varepsilon_b} - (i\hbar)^{-2}(\Omega_a, \Omega_b)_\Omega + \frac{1}{2}(i\hbar)^{-1}[\Omega_{ab}, \Omega] = 0, \tag{3.19}$$

where Ω_{ab} in general is a ϕ^a-dependent operator with ghost number minus two. From (3.19) one may then derive integrability conditions for Ω_{ab} which in turn introduces an operator Ω_{abc} with ghost number minus three, when the Ω-commutator is divided out. Thus, Y_a is replaced by a whole set of operators, and the integrability condition (3.3) for Y_a is replaced by a whole set of integrability conditions for these operators. These integrability conditions may be viewed as generalized Maurer-Cartan equations. In fact, for Lie groups we may choose

$$\Omega_a = \lambda_a^b(\phi)\mathcal{P}_b, \quad [\Omega_a, \Omega_b] = 0, \quad \Omega_{ab} = 0, \tag{3.20}$$

where $\lambda_a^b(\phi)$ only depends on ϕ^a. In this case (3.19) reduces to

$$\partial_a\lambda_b^c - \partial_b\lambda_a^c(-1)^{\varepsilon_a\varepsilon_b} = \lambda_a^e\lambda_b^d U_{de}^c(-1)^{\varepsilon_b\varepsilon_e+\varepsilon_c+\varepsilon_d+\varepsilon_e}, \quad \lambda_a^b(0) = \delta_a^b, \tag{3.21}$$

which are the classical Maurer-Cartan equations. In fact, (3.20) is also valid for quasigroup first rank theories [4]. We expect a nonzero Ω_{ab} to be necessary only for theories of rank two and higher. That a weaker form of Maurer-Cartan equations is necessary for rank two and higher should be connected to the fact that $[\Omega, \mathcal{P}_a]$ then no longer satisfy a closed algebra. However, notice that a nonzero Ω_{ab} is possible even in Lie group theories due to the ambiguity (3.14). The Maurer-Cartan equations (3.19) only retain their form under the replacement (3.14) if we at the same time make the replacement

$$\begin{aligned}\Omega_{ab} \to \quad & \Omega_{ab} + \Big(2(K_a\overleftarrow{\partial}_b - K_b\overleftarrow{\partial}_a(-1)^{\varepsilon_a\varepsilon_b}) - \\ & -(i\hbar)^{-2}((K_a, \Omega_b)_\Omega - (K_b, \Omega_a)_\Omega(-1)^{\varepsilon_a\varepsilon_b})\Big)(-1)^{\varepsilon_a+\varepsilon_b}.\end{aligned} \tag{3.22}$$

4 The quantum master equation

In [4] we proposed a quantum master equation for all the integrability conditions for the Ω-operators that follow from (3.3) and (3.11). All the Ω-operators were there imbedded in one master charge S. It is given by

$$\begin{aligned}
&S(\phi,\eta) \equiv G + \eta^a\Omega_a(\phi) + \frac{1}{2}\eta^b\eta^a\Omega_{ab}(\phi)(-1)^{\varepsilon_b} + \\
&+\frac{1}{6}\eta^c\eta^b\eta^a\Omega_{abc}(\phi)(-1)^{\varepsilon_b+\varepsilon_a\varepsilon_c} + \ldots \\
&\ldots + \frac{1}{n!}\eta^{a_n}\cdots\eta^{a_1}\Omega_{a_1\cdots a_n}(\phi)(-1)^{\varepsilon_n} + \ldots, \\
&\varepsilon_n \equiv \sum_{k=1}^{[\frac{n}{2}]}\varepsilon_{a_{2k}} + \sum_{k=1}^{[\frac{n-1}{2}]}\varepsilon_{a_{2k-1}}\varepsilon_{a_{2k+1}},
\end{aligned} \tag{4.1}$$

where G is the ghost charge operator defined in (2.11). The variables η^a, $\varepsilon(\eta^a) = \varepsilon_a + 1$, are new parameters which may be viewed as superpartners to ϕ^a. The general sign factor $(-1)^{\varepsilon_n}$, which is different from the one in [4], makes the Ω-operators have the symmetry properties

$$\begin{aligned}
&n \text{ even or odd}: \quad \Omega_{\cdots a_{2k}a_{2k+1}\cdots} = -\Omega_{\cdots a_{2k+1}a_{2k}\cdots}(-1)^{\varepsilon_{a_{2k-1}}\varepsilon_{a_{2k}}+\varepsilon_{a_{2k}}\varepsilon_{a_{2k+1}}+\varepsilon_{a_{2k+1}}\varepsilon_{a_{2k-1}}}, \\
&n \text{ even}: \quad \Omega_{\cdots a_{2k+1}a_{2k+2}\cdots} = -\Omega_{\cdots a_{2k+2}a_{2k+1}\cdots}(-1)^{\varepsilon_{a_{2k-1}}\varepsilon_{a_{2k+1}}+\varepsilon_{a_{2k+1}}\varepsilon_{a_{2k+2}}+\varepsilon_{a_{2k+2}}\varepsilon_{a_{2k-1}}}, \\
&n \text{ odd}: \quad \Omega_{\cdots a_{2k-1}a_{2k}\cdots} = -\Omega_{\cdots a_{2k}a_{2k-1}\cdots}(-1)^{\varepsilon_{a_{2k-1}}\varepsilon_{a_{2k}}+\varepsilon_{a_{2k}}\varepsilon_{a_{2k+1}}+\varepsilon_{a_{2k+1}}\varepsilon_{a_{2k-1}}}.
\end{aligned} \tag{4.2}$$

In addition the Ω-operators satisfy the properties

$$\varepsilon(\Omega_{a_1\cdots a_n}) = \varepsilon_{a_1} + \ldots + \varepsilon_{a_n} + n, \quad [G, \Omega_{a_1\cdots a_n}] = -ni\hbar\Omega_{a_1\cdots a_n}. \tag{4.3}$$

The last relation implies that $\Omega_{a_1\cdots a_n}$ has ghost number minus n. If we assign ghost number one to η^a, generalizing the original ghost number, then S has ghost number zero. The main conjecture in [4] was that the integrability conditions (3.19) together with those that follow from (3.19) are contained in the quantum master equation

$$(S,S)_\Delta = i\hbar[\Delta, S], \tag{4.4}$$

where Δ is the extended nilpotent BFV-BRST charge operator given by

$$\Delta \equiv \Omega - i\hbar\eta^a\partial_a, \quad \Delta^2 = 0, \tag{4.5}$$

Δ is odd and has extended ghost number one. The quantum antibracket $(S,S)_\Delta$ is defined by (A.1) in appendix A with Q replaced by Δ. In terms of commutators it is

$$(S,S)_\Delta \equiv [[S,\Delta],S]. \tag{4.6}$$

The master equation (4.4) may therefore be written as

$$[S,[S,\Delta]] = i\hbar[S,\Delta]. \tag{4.7}$$

The explicit form of $[S,\Delta]$ is to the lowest orders in η^a

$$\begin{aligned}
&[S,\Delta] = i\hbar\Omega + \eta^a[\Omega_a,\Omega] + \eta^b\eta^a\Omega_a \overleftarrow{\partial_b}\, i\hbar(-1)^{\varepsilon_b} + \frac{1}{2}\eta^b\eta^a[\Omega_{ab},\Omega](-1)^{\varepsilon_b} + \\
&+\frac{1}{2}\eta^c\eta^b\eta^a\Omega_{ab}\overleftarrow{\partial_c}\, i\hbar(-1)^{\varepsilon_b+\varepsilon_c} + \frac{1}{6}\eta^c\eta^b\eta^a[\Omega_{abc},\Omega](-1)^{\varepsilon_b+\varepsilon_a\varepsilon_c} + O(\eta^4).
\end{aligned} \tag{4.8}$$

When this is inserted into (4.7) we find that the quantum master equation is satisfied identically to zeroth and first order in η^a. To second order in η^a it yields exactly (3.19), and to the third order in η^a it yields

$$\begin{aligned}
&\partial_a\Omega_{bc}(-1)^{\varepsilon_a\varepsilon_c}+\frac{1}{2}(i\hbar)^{-2}(\Omega_a,\Omega_{bc})_\Omega(-1)^{\varepsilon_a\varepsilon_c}+cycle(a,b,c)=\\
&=-(i\hbar)^{-3}(\Omega_a,\Omega_b,\Omega_c)_\Omega(-1)^{\varepsilon_a\varepsilon_c}-\frac{2}{3}(i\hbar)^{-1}[\Omega'_{abc},\Omega],\\
&\Omega'_{abc}\equiv\Omega_{abc}-\frac{1}{8}\Big\{(i\hbar)^{-1}[\Omega_{ab},\Omega_c](-1)^{\varepsilon_a\varepsilon_c}+cycle(a,b,c)\Big\}, \qquad (4.9)
\end{aligned}$$

where we have introduced the higher quantum antibracket of order 3 defined by (A.8) in appendix A with Q replaced by Ω, or equivalently by (A.11) in terms of the 2-antibracket. In fact, (4.9) coincides with the integrability conditions of (3.19). (Notice that if $\Omega_{ab}=\Omega_{abc}=0$, which we may have for rank one theories, Ω_a must satisfy the condition $(\Omega_a,\Omega_b,\Omega_c)_\Omega=0$. That this is satisfied we have checked up to quasigroups [4].)

The master equation (4.7) requires for consistency

$$[\Delta,(S,S)_\Delta]=0 \quad\Leftrightarrow\quad [\Delta,S]^2=0. \qquad (4.10)$$

This property was also checked in [4] up to the third order in η^a. To the second order it yields exactly (3.3).

Formally we may introduce conjugate momenta, π_a and ξ_a, to the variables ϕ^a and η^a. We have then

$$[\phi^a,\pi_b]=i\hbar\delta^a_b,\quad [\eta^a,\xi_b]=i\hbar\delta^a_b. \qquad (4.11)$$

All derivatives with respect to ϕ^a may then be replaced by π_a. The Lie equations (3.1)-(3.2) may *e.g.* be written as

$$\langle A(\phi)|(\pi_a-Y_a(\phi))=0,\quad [A(\phi),\pi_a-Y_a(\phi)]=0, \qquad (4.12)$$

and the Δ-operator (4.5) may be written as

$$\Delta\equiv\Omega+\eta^a\pi_a(-1)^{\varepsilon_a}. \qquad (4.13)$$

(This form was used in [4].) The use of ξ_a is so far unclear since we have not used any derivatives with respect to η^a. However, it allows us to define a total ghost charge by

$$\tilde{G}\equiv G-\frac{1}{2}\left(\xi_a\eta^a-\eta^a\xi_a(-1)^{\varepsilon_a}\right), \qquad (4.14)$$

where G is the ghost charge (2.11). In terms of $\tilde{G}$ we have

$$[\tilde{G},S]=0,\quad [\tilde{G},\Delta]=i\hbar\Delta. \qquad (4.15)$$

5 Formal properties of the quantum master equation

Let us define the transformed operators $S(\alpha)$ and $\Delta(\alpha)$ by

$$S(\alpha)\equiv e^{\frac{i}{\hbar}\alpha F}Se^{-\frac{i}{\hbar}\alpha F},\quad \Delta(\alpha)\equiv e^{\frac{i}{\hbar}\alpha F}\Delta e^{-\frac{i}{\hbar}\alpha F}, \qquad (5.1)$$

where α is a real parameter and F an arbitrary even operator with total ghost number zero since we require $S(\alpha)$ and $\Delta(\alpha)$ to have total ghost number zero and one. If S and Δ satisfy the master equation (4.4), then it is easily seen that $S(\alpha)$ and $\Delta(\alpha)$ satisfy the transformed master equation

$$(S(\alpha), S(\alpha))_{\Delta(\alpha)} = i\hbar[\Delta(\alpha), S(\alpha)]. \tag{5.2}$$

If we restrict F in (5.1) to satisfy the master equation (4.4), *i.e.*

$$(F, F)_{\Delta} = i\hbar[\Delta, F], \tag{5.3}$$

then $\Delta(\alpha)$ in (5.1) reduces to

$$\Delta(\alpha) = \Delta + (i\hbar)^{-1}[\Delta, F](1 - e^{-\alpha}). \tag{5.4}$$

(Notice that (5.3) implies $\Delta''(\alpha) + \Delta'(\alpha) = 0$.) For $F = S$ we have then that S satisfies the master equation (4.4) with Δ replaced by $\Delta(\alpha)$ in (5.4) where $F = S$.

There are also transformations on S leaving Δ unaffected for which the master equation (4.4) is invariant. From (5.1) this is the case if $\Delta(\alpha) = \Delta$ which requires

$$[\Delta, F] = 0. \tag{5.5}$$

In order for the transformed S to be in the form (4.1), F should not depend on π_a and ξ_a in (4.11). If we assume that $F(\phi, \eta)$ may be given by a power series in ϕ^a and η^a we find that (5.5) implies (the proof is given in appendix B)

$$F(\phi, \eta) = F_0 + (i\hbar)^{-1}[\Delta, \Psi(\phi, \eta)], \tag{5.6}$$

where $F_0 = F(0,0)$ and $[\Delta, \Psi(\phi, \eta)]|_{\phi=\eta=0} = 0$. Ψ is an odd operator with total ghost number minus one which does not depend on π_a and ξ_a. (It has the form (B.10) in appendix B.) An interesting consequence of the result (5.6) is that the extended BRST singlets are identical to the original BRST singlets under Ω since $[\Delta, F_0] = [\Omega, F_0] = 0$. Now although the general invariance transformation on S is given by (5.1) with the F-operator (5.6), it is natural and consistent to set $F_0 = 0$ in which case we find the following class of invariance transformations [8]

$$S \rightarrow S' \equiv \exp\Big\{-(i\hbar)^{-2}[\Delta, \Psi]\Big\} S \exp\Big\{(i\hbar)^{-2}[\Delta, \Psi]\Big\}. \tag{5.7}$$

This we view as the natural automorphism of the master equation (4.4). These transformations leave the $\phi^a = \eta^a = 0$ component of S invariant in a trivial manner since $[\Delta, \Psi(\phi, \eta)]|_{\phi=\eta=0} = 0$. The infinitesimal invariance transformations and their properties which follow from (5.7) are

$$\begin{aligned}
&\delta S = (i\hbar)^{-2}[S, [\Delta, \Psi]] = (i\hbar)^{-2}\Big((S, \Psi)_{\Delta} - \frac{1}{2}[\Delta, [\Psi, S]]\Big),\\
&\delta_{21} S \equiv (\delta_2\delta_1 - \delta_1\delta_2)S = (i\hbar)^{-2}[S, [\Delta, \Psi_{21}]] = (i\hbar)^{-2}\Big((S, \Psi_{21})_{\Delta} - \frac{1}{2}[\Delta, [\Psi_{21}, S]]\Big),\\
&\Psi_{21} = (i\hbar)^{-2}(\Psi_2, \Psi_1)_{\Delta}.
\end{aligned} \tag{5.8}$$

Since $S = G$ is a trivial solution of the master equation (4.4), we find from (5.7) the following expression for the master charge S in (4.1)

$$S = \exp\left\{-(i\hbar)^{-2}[\Delta, \Psi]\right\} G \exp\left\{(i\hbar)^{-2}[\Delta, \Psi]\right\}, \tag{5.9}$$

where Ψ is the odd operator in (5.6) which has the form (B.10) in appendix B. This is the general solution of the quantum master equation (4.4) within the class connected by the transformations (5.7). The transformations (5.7) act transitivily on (5.9).

6 Extended Lie equations

The invariance transformations of the quantum master equation (4.4) which follow from the transformation formulas (5.1) together with (5.5) suggest that we could define extended group elements by (cf.(3.7))

$$U(\phi, \eta) \equiv \exp\{\frac{i}{\hbar}F(\phi, \eta)\}, \quad [\Delta, F] = 0, \quad [\tilde{G}, F] = 0, \tag{6.1}$$

where $\tilde{G}$ is the extended ghost charge in (4.14). $F(\phi, \eta)$ is obviously given by (5.6). Interestingly enough $U(\phi) = U(\phi, \eta)|_{\eta=0}$ is equal to the general solution of (3.4) and (3.11) which provides for another argument in favor of the quantum master equation. Now if we also require $U(\phi, \eta)$ to be a unit transformation at $\phi^a = \eta^a = 0$ exactly as in (3.7) then the general solution is

$$U(\phi, \eta) \equiv \exp\left\{-(i\hbar)^{-2}[\Delta, \Psi(\phi, \eta)]\right\}, \tag{6.2}$$

which at $\eta^a = 0$ is identical to (3.8) if we make the identification $\Psi(\phi, \eta)|_{\eta=0} = \rho(\phi)$. By means of (6.2) we may then define transformed states and operators by

$$\langle\tilde{A}(\phi, \eta)| = \langle A|U^{-1}(\phi, \eta), \quad \tilde{A}(\phi, \eta) = U(\phi, \eta)AU^{-1}(\phi, \eta), \tag{6.3}$$

which at $\eta^a = 0$ are the group transformed states and operators satisfying the Lie equations (3.1)-(3.2) where the operator connections (3.4) have the form (3.11).

Also (6.3) are group transformed states and operators but under the extended group elements (6.2) which are parametrized by the supersymmetric pairs of parameters ϕ^a and η^a. The states and operators in (6.3) satisfy the extended Lie equations

$$\begin{aligned}
&\langle\tilde{A}(\phi, \eta)| \overleftarrow{\tilde{D}}_a \equiv \langle\tilde{A}(\phi, \eta)| \left(\overleftarrow{\partial}_a -(i\hbar)^{-1}\tilde{Y}_a(\phi, \eta)\right) = 0, \\
&\tilde{A}(\phi, \eta) \overleftarrow{\tilde{\nabla}}_a \equiv \tilde{A}(\phi, \eta) \overleftarrow{\partial}_a -(i\hbar)^{-1}[\tilde{A}(\phi, \eta), \tilde{Y}_a(\phi, \eta)] = 0,
\end{aligned} \tag{6.4}$$

where

$$\tilde{Y}_a(\phi, \eta) = i\hbar U(\phi, \eta)\left(U^{-1}(\phi, \eta) \overleftarrow{\partial}_a\right). \tag{6.5}$$

The expression (6.2) for $U(\phi, \eta)$ implies then that

$$\tilde{Y}_a(\phi, \eta) = (i\hbar)^{-1}[\Delta, \tilde{\Omega}_a(\phi, \eta)], \tag{6.6}$$

where

$$\tilde{\Omega}_a(\phi,\eta) =$$
$$= \int_0^1 d\alpha \exp\{-(i\hbar)^{-2}\alpha[\Delta,\Psi(\phi,\eta)]\}\left(\Psi(\phi,\eta)\overleftarrow{\partial_a}\right)\exp\{(i\hbar)^{-2}\alpha[\Delta,\Psi(\phi,\eta)]\}. \tag{6.7}$$

The general expression (5.9) for the master charge S implies in particular that the master charge S itself is a group transformed ghost charge under the extended group element (6.2) which means that it satisfies the Lie equation

$$S(\phi,\eta)\overleftarrow{\tilde{\nabla}}_a \equiv S(\phi,\eta)\overleftarrow{\partial_a} - (i\hbar)^{-1}[S(\phi,\eta),\tilde{Y}_a(\phi,\eta)] = 0. \tag{6.8}$$

This equation together with the representation (6.6) of $\tilde{Y}_a(\phi,\eta)$ may be used to resolve $\tilde{\Omega}_a(\phi,\eta)$ in terms of S. We find from (6.8) to the lowest order in η^a

$$\tilde{\Omega}_a(\phi,\eta) = \Omega_a(\phi) + \frac{1}{2}\eta^b\left(\Omega_{ba}(-1)^{\varepsilon_a} + (i\hbar)^{-1}[\Omega_b,\Omega_a]\right) + \dots \tag{6.9}$$

However, this solution is not unique. The arbitrariness is

$$\tilde{\Omega}_a \to \tilde{\Omega}_a + (i\hbar)^{-1}[\Delta,\tilde{K}_a], \tag{6.10}$$

which corresponds to the arbitrariness in (3.14) and (3.22). (The $\eta^a \to 0$ limit yields (3.14).) The relation between the extended group transformation and the nonextended one may also be expressed in terms of the master charge S. We have

$$|\tilde{A}(\phi,\eta)\rangle = V(\phi,\eta)|A(\phi)\rangle, \quad \tilde{A}(\phi,\eta) = V(\phi,\eta)A(\phi)V^{-1}(\phi,\eta), \tag{6.11}$$

where

$$V(\phi,\eta) = U(\phi,\eta)U^{-1}(\phi). \tag{6.12}$$

Since

$$i\hbar\alpha\frac{d}{d\alpha}U(\phi,\alpha\eta) = i\hbar\eta^a\frac{\partial}{\partial\eta^a}U(\phi,\alpha\eta) = [G,U(\phi,\eta)] = (S(\phi,\alpha\eta)-G)U(\phi,\alpha\eta), \tag{6.13}$$

we get

$$V(\phi,\eta) = T_\alpha \exp\left\{(i\hbar)^{-1}\int_0^1 d\alpha\frac{1}{\alpha}(S(\phi,\alpha\eta)-G)\right\} = 1 + (i\hbar)^{-1}\eta^a\Omega_a +$$
$$+(i\hbar)^{-1}\frac{1}{4}\eta^a\eta^b\left(\Omega_{ba}(-1)^{\varepsilon_a} + (i\hbar)^{-1}(\Omega_b\Omega_a + \Omega_a\Omega_b(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)})\right) + \dots \tag{6.14}$$

7 Some further properties of the extended states and operators

Here we give some additional properties and interpretations of the results in section 6. First we notice that if the original states and operators are eigenstates and eigenoperators to the ghost charge G then the transformed states and operators in (6.3) are eigenstates and eigenoperators to the master charge S, *i.e.*

$$S|\tilde{A}\rangle_g = i\hbar g|\tilde{A}\rangle_g, \quad [S, \tilde{A}_g] = i\hbar g\tilde{A}_g. \tag{7.1}$$

Another property which follows from the representation (6.3) is

$$\begin{aligned} \Omega|A\rangle = 0 \quad &\Rightarrow \quad \Delta|\tilde{A}(\phi,\eta)\rangle = 0, \\ [\Omega, A] = 0 \quad &\Rightarrow \quad [\Delta, \tilde{A}(\phi,\eta)] = 0, \end{aligned} \tag{7.2}$$

i.e. if the original states and operators are invariant under the BRST charge Ω, then the extended states and operators are invariant under the extended BRST charge Δ. Δ should of course be hermitian.

The extended BRST charge Δ may be identified with the conventional BRST charge if ϕ^a are identified with the Lagrange multipliers and if ξ_a and η^a are identified with antighosts and their conjugate momenta. Although the group element $U(\phi,\eta)$ should be unitary, there is an interesting interpretation of the extended states

$$|\tilde{A}\rangle \equiv U(\phi,\eta)|A\rangle, \tag{7.3}$$

when $U(\phi,\eta)$ is *hermitian*. If we choose $|A\rangle$ to be independent of the conjugate momenta $\mathcal{P}_a$ to the ghosts, *i.e.*

$$\mathcal{C}^a|A\rangle = 0, \tag{7.4}$$

then $\Omega|A\rangle = 0$ and $\Delta|A\rangle = 0$ which imply $\Delta|\tilde{A}\rangle = 0$. In fact, the states in (7.3) are not only BRST invariant under Δ but they are also formal inner product states [9]. A *hermitian* choice of Ψ of the form $\Psi \propto \mathcal{P}_a\phi^a$ in the representation (6.2) of U in (7.3) leads to inner product states with group geometric properties. In [10] such states were considered for Lie group theories.

Acknowledgments

I.A.B. would like to thank Lars Brink for his very warm hospitality at the Institute of Theoretical Physics, Chalmers and Göteborg University. The work of I.A.B. is supported by INTAS grant 96-0308 and by RFBR grants 99-01-00980, 99-02-17916. I.A.B. and R.M. are thankful to the Royal Swedish Academy of Sciences for financial support.

Appendix A

Quantum antibrackets.

Quantum antibrackets are algebraic tools like commutators which should be useful in any quantum theory in which there are fundamental odd operators like in BRST-quantization of gauge theories and supersymmetric theories. So far they have only been applied to gauge theories. They provide for an operator formulation of the BV-quantization [5, 8, 7]. Here as in [4, 7] they are applied to the BFV-BRST quantization.

The basic quantum antibracket is defined by [5]

$$(f,g)_Q \equiv \frac{1}{2}\Big([f,[Q,g]] - [g,[Q,f]](-1)^{(\varepsilon_f+1)(\varepsilon_g+1)}\Big), \tag{A.1}$$

where f and g are any operators with Grassmann parities $\varepsilon(f) \equiv \varepsilon_f$ and $\varepsilon(g) \equiv \varepsilon_g$ respectively. Q is an odd operator, $\varepsilon(Q) = 1$. The commutators on the right-hand side is the graded commutator (2.5). The quantum antibracket (A.1) satisfies the properties:

1) Grassmann parity

$$\varepsilon((f,g)_Q) = \varepsilon_f + \varepsilon_g + 1. \tag{A.2}$$

2) Symmetry

$$(f,g)_Q = -(g,f)_Q(-1)^{(\varepsilon_f+1)(\varepsilon_g+1)}. \tag{A.3}$$

3) Linearity

$$(f+g,h)_Q = (f,h)_Q + (g,h)_Q, \quad (\text{for } \varepsilon_f = \varepsilon_g). \tag{A.4}$$

4) If one entry is an odd/even parameter λ we have

$$(f,\lambda)_Q = 0 \quad \text{for any operator } f. \tag{A.5}$$

5) the generalized Jacobi identities (This general form was given in [7])

$$\begin{aligned}
&(f,(g,h)_Q)_Q(-1)^{(\varepsilon_f+1)(\varepsilon_h+1)} + cycle(f,g,h) = \\
&= \frac{1}{6}(-1)^{\varepsilon_f+\varepsilon_g+\varepsilon_h}\Big\{\Big([f,[g,[h,Q^2]]] + \frac{1}{2}[[f,[g,[h,Q]]],Q]\Big)(-1)^{\varepsilon_f\varepsilon_h} + \\
&+ \Big([f,[h,[g,Q^2]]] + \frac{1}{2}[[f,[h,[g,Q]]],Q]\Big)(-1)^{\varepsilon_h(\varepsilon_f+\varepsilon_g)}\Big\} + cycle(f,g,h),
\end{aligned} \tag{A.6}$$

6) and the generalized Leibniz rule

$$\begin{aligned}
&(fg,h)_Q - f(g,h)_Q - (f,h)_Q g(-1)^{\varepsilon_g(\varepsilon_h+1)} = \\
&= \frac{1}{2}\Big([f,h][g,Q](-1)^{\varepsilon_h(\varepsilon_g+1)} + [f,Q][g,h](-1)^{\varepsilon_g}\Big).
\end{aligned} \tag{A.7}$$

The properties 1)-4) agree exactly with the corresponding properties of the classical antibracket (f,g) for functions f and g. However, the classical antibracket satisfies in addition 6) and 7) where the right-hand side is zero. The right-hand side of (A.7) is zero if we confine ourselves to a maximal set of commuting operators. They may *e.g.* be functions of a basic set of commuting coordinate operators. The right-hand side of (A.6) is then zero if Q and Q^2 are at most quadratic in the conjugate momenta to these coordinate operators. The classical antibracket is then finally obtained in the coordinate representation.

The quantum antibracket (A.1) is also useful for arbitrary operators. However, in this case we need further algebraic tools represented by higher order quantum antibrackets defined by [5, 7]

$$\begin{aligned}
&(f_{a_1},\dots,f_{a_n})_Q \equiv -\frac{1}{n!}(-1)^{E_n}\sum_{\text{sym}}[\cdots[[Q,f_{a_1}],\cdots,f_{a_n}] \equiv \\
&\equiv -\frac{1}{n!}[\cdots[[Q,f_{b_1}],\cdots,f_{b_n}]\lambda^{b_n}\cdots\lambda^{b_1}\overleftarrow{\partial}_{a_1}\overleftarrow{\partial}_{a_2}\cdots\overleftarrow{\partial}_{a_n}(-1)^{E_n}, \\
&E_n \equiv \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\varepsilon_{a_{2k+1}}.
\end{aligned} \tag{A.8}$$

They satisfy the properties

$$
\begin{aligned}
&(\ldots, f_a, f_b, \ldots)_Q = -(-1)^{(\varepsilon_a+1)(\varepsilon_b+1)}(\ldots, f_b, f_a, \ldots)_Q,\\
&\varepsilon((f_{a_1}, f_{a_2}, \ldots, f_{a_n})_Q) = \varepsilon_{a_1} + \cdots + \varepsilon_{a_n} + 1.
\end{aligned}
\tag{A.9}
$$

The higher order quantum antibrackets may also be expressed recursively in terms of the next lower ones. We have [7]

$$
\begin{aligned}
&(f_{a_1}, \ldots, f_{a_n})_Q = \frac{1}{n}\sum_{k=1}^{n}[(f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_{k+1}}, \ldots, f_{a_n})_Q, f_{a_k}](-1)^{B_{k,n}},\\
&B_{k,n} \equiv \varepsilon_{a_k}(\varepsilon_{a_{k+1}} + \ldots + \varepsilon_{a_n}) + \sum_{s=2[\frac{k}{2}]+1}^{n} \varepsilon_{a_s}.
\end{aligned}
\tag{A.10}
$$

To the lowest order we have explicitly

$$
(f_a, f_b, f_c)_Q = \frac{1}{3}(-1)^{(\varepsilon_a+1)(\varepsilon_c+1)}\Big([(f_a, f_b)_Q, f_c](-1)^{\varepsilon_c+(\varepsilon_a+1)(\varepsilon_c+1)} + cycle(a, b, c)\Big).
\tag{A.11}
$$

The higher quantum antibrackets satisfy the following generalized Jacobi identities [7]

$$
\begin{aligned}
&\sum_{k=1}^{n}(f_{a_k}, (f_{a_1}, \ldots, f_{a_{k-1}}, f_{a_{k+1}}, \ldots, f_{a_n})_Q)_Q(-1)^{D_{k,n}} = -\frac{n-2}{2}[Q, (f_{a_1}, \ldots, f_{a_n})_Q] -\\
&-R_n + (-1)^{E_n}\frac{1}{n!}\sum_{\text{sym}}[\cdots[[Q^2, f_{a_1}], f_{a_2}], \cdots, f_{a_n}] = 0,
\end{aligned}
\tag{A.12}
$$

where

$$
\begin{aligned}
&R_n \equiv \frac{1}{2}\sum_{k=2}^{n-2}\sum_{\text{sym}}[(f_{a_1}, \ldots, f_{a_k})_Q, (f_{a_{k+1}}, \ldots, f_{a_n})_Q](-1)^{F_{k,n}},\\
&F_{k,n} \equiv \sum_{r=1}^{(n,k)} \varepsilon_{a_r},
\end{aligned}
\tag{A.13}
$$

where $(n,k) \equiv n$ for k odd, and $(n,k) \equiv k$ for k even. The symmetrized sum is over all different orders with additional sign factors $(-)^{E_n+\tilde{E}_n+A_n}$ where $\tilde{E}_n$ is E_n for the new order and A_n from the reordering of the monomial $\lambda^{a_1}\cdots\lambda^{a_n}$. For $n=3$ (A.12) reduces to (A.6).

Appendix B

Proof of (5.6).

Following section 5 we let $F(\phi,\eta)$ be an operator which neither depends on π_a nor ξ_a and which is given by a power expansion in ϕ^a and η^a. We want to solve the condition (5.5), *i.e.*

$$
[\Delta, F(\phi,\eta)] = 0.
\tag{B.1}
$$

To this purpose we introduce the operators

$$\Lambda \equiv \eta^a \pi_a (-1)^{\varepsilon_a}, \quad \tilde{\Lambda} \equiv \xi_a \phi^a (-1)^{\varepsilon_a}, \tag{B.2}$$

with the properties

$$\varepsilon(\Lambda) = \varepsilon(\tilde{\Lambda}) = 1, \quad \Lambda^2 = \tilde{\Lambda}^2 = 0. \tag{B.3}$$

Λ has total ghost charge plus one and $\tilde{\Lambda}$ minus one. We have then

$$\Delta = \Omega + \Lambda, \quad N \equiv (i\hbar)^{-1}[\Lambda, \tilde{\Lambda}] = \pi_a \phi^a + \xi_a \eta^a, \quad [N, \Lambda] = [N, \tilde{\Lambda}] = 0, \tag{B.4}$$

from which it follows that

$$(i\hbar)^{-1}[N, F(\phi, \eta)] = -\left(\phi^a \frac{\partial}{\partial \phi^a} + \eta^a \frac{\partial}{\partial \eta^a}\right) F(\phi, \eta). \tag{B.5}$$

By commuting (B.1) with $\tilde{\Lambda}$ and taking (B.4), (B.5) into account we get

$$\left(\phi^a \frac{\partial}{\partial \phi^a} + \eta^a \frac{\partial}{\partial \eta^a}\right) F(\phi, \eta) = (i\hbar)^{-2}[[F(\phi, \eta), \tilde{\Lambda}], \Delta]. \tag{B.6}$$

Next, let us make the following canonical transformation in (B.6):

$$\begin{aligned} &\phi^a \to \alpha \phi^a, \quad \pi_a \to \alpha^{-1} \pi_a, \\ &\eta^a \to \alpha \eta^a, \quad \xi_a \to \alpha^{-1} \xi_a, \end{aligned} \tag{B.7}$$

where α is a parameter. Then we get

$$\alpha \frac{d}{d\alpha} F(\alpha\phi, \alpha\eta) = (i\hbar)^{-2}[[F(\alpha\phi, \alpha\eta), \tilde{\Lambda}], \Delta]. \tag{B.8}$$

Integration over α yields then

$$\begin{aligned} &F(\phi, \eta) = F(0,0) + (i\hbar)^{-2}[\int_0^1 \frac{d\alpha}{\alpha}[F(\alpha\phi, \alpha\eta), \tilde{\Lambda}], \Delta] \equiv F(0,0) + (i\hbar)^{-1}[\Delta, \Psi], \\ &[\Delta, \Psi]|_{\phi=\eta=0} = 0, \end{aligned} \tag{B.9}$$

which is the assertion in (5.6). Notice that Ψ has the explicit form

$$\Psi(\phi, \eta) = (i\hbar)^{-1} \int_0^1 \frac{d\alpha}{\alpha}[F(\alpha\phi, \alpha\eta), \tilde{\Lambda}] + (i\hbar)^{-1}[\Delta, K(\phi, \eta)], \tag{B.10}$$

where $K(\phi, \eta)$ is an arbitrary operator with total ghost number minus two and which neither depends on π_a nor ξ_a. Notice that

$$\Psi(\phi, \eta)|_{\eta=0} = \int_0^1 d\alpha F_a(\alpha\phi)(-1)^{\varepsilon_a} + (i\hbar)^{-1}[\Omega, K(\phi, 0)], \tag{B.11}$$

where $F_a(\phi) = \left(F(\phi, \eta)\frac{\overleftarrow{\partial}}{\partial \eta^a}\right)\Big|_{\eta=0}$. (B.11) agrees exactly with (3.16) with the identifications

$$\rho(\phi) = \Psi(\phi, \eta)|_{\eta=0}, \quad G_a(\phi) = F_a(\phi)(-1)^{\varepsilon_a}, \quad \kappa(\phi) = K(\phi, 0). \tag{B.12}$$

The integrability conditions (3.17) of $G_a(\phi)$ are encoded in (B.9).

Appendix C

A generalization.

Consider the Lie equations (3.1)-(3.2). If we require the connection operator Y_a to only satisfy $[\Omega, Y_a] = 0$, then we have

$$Y_a = Y_a^{(S)} + (i\hbar)^{-1}[\Omega, \Omega_a], \tag{C.1}$$

where $Y_a^{(S)}$ is the BRST singlet part of Y_a. The integrability conditions (3.3) imply then

$$\begin{aligned}
&Y_a^{(S)} \overleftarrow{\partial_b} - Y_b^{(S)} \overleftarrow{\partial_a} (-1)^{\varepsilon_a\varepsilon_b} - (i\hbar)^{-1}[Y_a^{(S)}, Y_b^{(S)}] + \\
&+(i\hbar)^{-1}[\Omega, \left(\Omega_a \overleftarrow{\partial_b} - (i\hbar)^{-1}[\Omega_a, Y_b^{(S)}]\right) - \\
&- \left(\Omega_b \overleftarrow{\partial_a} (-1)^{\varepsilon_a\varepsilon_b} - (i\hbar)^{-1}[\Omega_b, Y_a^{(S)}]\right)(-1)^{\varepsilon_a\varepsilon_b} - (i\hbar)^{-2}(\Omega_a, \Omega_b)_\Omega] = 0.
\end{aligned} \tag{C.2}$$

These conditions are natural to split as follows

$$\begin{aligned}
&Y_a^{(S)} \overleftarrow{\partial_b} - Y_b^{(S)} \overleftarrow{\partial_a} (-1)^{\varepsilon_a\varepsilon_b} - (i\hbar)^{-1}[Y_a^{(S)}, Y_b^{(S)}] = 0, \\
&\Omega_a \overleftarrow{\nabla}_b^{(S)} - \Omega_b \overleftarrow{\nabla}_a^{(S)} (-1)^{\varepsilon_a\varepsilon_b} - (i\hbar)^{-2}(\Omega_a, \Omega_b)_\Omega = -\frac{1}{2}(i\hbar)^{-1}[\Omega_{ab}, \Omega],
\end{aligned} \tag{C.3}$$

where

$$\overleftarrow{\nabla}_a^{(S)} \equiv \overleftarrow{\partial_a} - (i\hbar)^{-1}[\,\cdot\,, Y_a^{(S)}], \quad [\overleftarrow{\nabla}_a^{(S)}, \overleftarrow{\nabla}_b^{(S)}] = 0. \tag{C.4}$$

Since the Lie equations (3.1)-(3.2) may be written as

$$\langle A(\phi)| \left(\overleftarrow{\nabla}_a^{(S)} - (i\hbar)^{-1} Y_a(\phi)\right) = 0, \quad A(\phi) \overleftarrow{\nabla}_a^{(S)} - (i\hbar)^{-1}[A(\phi), Y_a(\phi)] = 0, \tag{C.5}$$

where $Y_a(\phi) \equiv (i\hbar)^{-1}[\Omega, \Omega_a]$, it follows that we still have the quantum master equation (4.4) but here with the Δ-operator defined by

$$\Delta \equiv \Omega + \eta^a(\pi_a - Y_a^{(S)})(-1)^{\varepsilon_a}. \tag{C.6}$$

In this case we find the general group element $\tilde{U}(\phi, \eta) = U^{(S)}(\phi, \eta)U(\phi, \eta)$. $Y_a^{(S)}$ is expressed in terms of $U^{(S)}(\phi, \eta)$ and $U(\phi, \eta)$ is given by (6.3) in section 6. Thus, the singlet part of Y_a may be transformed away.

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